ON THE DERIVATION OF TWO-DIMENSIONAL EQUATIONS IN THE THEORY OF THIN ELASTIC PLATES

(K POSTROENIIU DVUMERNYKH URAVNENII TEORII UPRUGIKH TONKIKH PLASTINOK)

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A.L. GOL'DENVEIZER and A.V. KOLOS
(Moscow)

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This paper deals with the methods of constructing an approximate theory of thin elastic plates without using the assumptions of the type of Kirchhoff's hypothesis. Until recently, the only method of solution of this problem was one based on the application of power series or series arranged in Legendre polynomials. Lately, however, some papers have appeared in which the same purpose is achieved by means of asymptotic integration of the equations of the theory of elasticity. In the present study the properties of these methods are discussed and the equations, which arise in the application of the asymptotic method to the problem of general deformation of a thin plate whose middle plane is referred to an arbitrary orthogonal system of curvilinear coordinates, are derived.

1. We refer the middle plane of a plate to an orthogonal system of curvilinear coordinates α , β and assume that the γ-coordinate is perpendicular to the middle plane. Then the differential equations of the three-dimensional problem of the theory of elasticity will have the form

$$H_\alpha \frac{\partial \sigma_{\alpha\alpha}}{\partial \alpha} + H_\beta \frac{\partial \sigma_{\alpha\beta}}{\partial \beta} + \frac{\partial \sigma_{\alpha\gamma}}{\partial \gamma} - H_\alpha \frac{\partial \ln H_\beta}{\partial \alpha} (\sigma_{\alpha\alpha} - \sigma_{\beta\beta}) - 2H_\beta \frac{\partial \ln H_\alpha}{\partial \beta} \sigma_{\alpha\beta} = 0 \qquad (\alpha\beta)$$

$$H_\alpha \frac{\partial \sigma_{\alpha\gamma}}{\partial \alpha} + H_\beta \frac{\partial \sigma_{\beta\gamma}}{\partial \beta} + \frac{\partial \sigma_{\gamma\gamma}}{\partial \gamma} - H_\alpha \frac{\partial \ln H_\beta}{\partial \alpha} \sigma_{\alpha\gamma} - H_\beta \frac{\partial \ln H_\alpha}{\partial \beta} \sigma_{\beta\gamma} = 0$$

$$E\left(H_\alpha \frac{\partial u_\alpha}{\partial \alpha} - H_\beta \frac{\partial \ln H_\alpha}{\partial \beta} u_\beta\right) = \sigma_{\alpha\alpha} - \nu (\sigma_{\beta\beta} + \sigma_{\gamma\gamma}) \qquad (\alpha\beta)$$

$$E \frac{\partial W}{\partial \gamma} = \sigma_{\gamma\gamma} - \nu (\sigma_{\alpha\alpha} + \sigma_{\beta\beta}) \qquad (1.1)$$

$$E\left(H_\alpha \frac{\partial u_\beta}{\partial \alpha} + H_\beta \frac{\partial u_\alpha}{\partial \beta} + H_\alpha \frac{\partial \ln H_\beta}{\partial \alpha} u_\beta + H_\beta \frac{\partial \ln H_\alpha}{\partial \beta} u_\alpha\right) = 2(1+\nu)\sigma_{\alpha\beta}$$

$$E\left(H_\alpha \frac{\partial W}{\partial \alpha} + \frac{\partial u_\alpha}{\partial \gamma}\right) = 2(1+\nu)\sigma_{\alpha\gamma} \qquad (\alpha\beta)$$

Here $\sigma_{\alpha\alpha}$, $\sigma_{\alpha\beta}$, $\sigma_{\alpha\gamma}$, $\sigma_{\beta\beta}$, $\sigma_{\beta\gamma}$, $\sigma_{\gamma\gamma}$ are components of the stress tensor, u_α, u_β, W are components of the displacement vector, H_α, H_β are Lamé's parameters, E , ν are, respectively, the modulus of elasticity and Poisson's ratio. The symbol $(\alpha\beta)$ here and in what follows indicates that there exists the second relation which is obtained from the given one by the interchange of the indices α and β .

Assuming that the plate is loaded in an arbitrary manner on the upper and the lower plane,we have the conditions on the surfaces $\gamma = \pm h$

$$\begin{aligned} \sigma_{\gamma\gamma} &= a(\alpha, \beta), \quad \sigma_{\alpha\gamma} = h^{-1}c_\alpha(\alpha, \beta) \quad (\alpha\beta) \quad \text{for } \gamma = h \\ \sigma_{\gamma\gamma} &= b(\alpha, \beta), \quad \sigma_{\alpha\gamma} = h^{-1}d_\alpha(\alpha, \beta) \quad (\alpha\beta) \quad \text{for } \gamma = -h \end{aligned} \tag{1.2}$$

The conditions on the side surfaces of the plate will be formulated later.

The state of stress of the plate can be represented as a sum of the symmetric and the skew-symmetric states of stress.

The symmetric state of stress is defined as one satisfying the conditions

$$\sigma_{\gamma\gamma} = \tfrac{1}{2} q(\alpha, \beta), \quad \sigma_{\alpha\gamma} = \pm \tfrac{1}{2} h^{-1} q_\alpha(\alpha, \beta) \quad (\alpha\beta) \quad \text{for } \gamma = \pm h \tag{1.3}$$

and such that

$$\begin{aligned} &\sigma_{\alpha\alpha},\ \sigma_{\alpha\beta},\ \sigma_{\beta\beta},\ \sigma_{\gamma\gamma},\ u_\alpha,\ u_\beta \ \text{ are even functions,} \\ &\sigma_{\alpha\gamma},\ \sigma_{\beta\gamma},\ W \ \text{ are odd functions} \end{aligned} \tag{1.4}$$

with respect to γ

The skew-symmetric state of stress is defined as one satisfying the conditions

$$\sigma_{\gamma\gamma} = \pm \tfrac{1}{2} p(\alpha, \beta), \quad \sigma_{\alpha\gamma} = \tfrac{1}{2} h^{-1} p_\alpha(\alpha, \beta) \quad (\alpha\beta) \quad \text{for } \gamma = \pm h \tag{1.5}$$

and such that

$$\begin{aligned} &\sigma_{\alpha\gamma},\ \sigma_{\beta\gamma},\ W \ \text{ are even functions,} \\ &\sigma_{\alpha\alpha},\ \sigma_{\alpha\beta},\ \sigma_{\beta\beta},\ \sigma_{\gamma\gamma},\ u_\alpha,\ u_\beta \ \text{ are odd functions} \end{aligned} \tag{1.6}$$

with respect to γ .

It is assumed that

$$p = a - b, \quad q = a + b, \quad q_\alpha = c_\alpha - d_\alpha \quad (\alpha\beta), \quad p_\alpha = c_\alpha + d_\alpha \quad (\alpha\beta)$$

The skew-symmetric state of stress corresponds to bending of the plate, in which p is the intensity of the surface load, p_α, p_β are the intensities of the surface moments; the symmetric state of stress corresponds to the state of generalized plane stress in which q_α, q_β are the intensities of loads parallel to the middle plane, and also to the state of compression of the plate in which q is the intensity of the compression load.

2. Let us write down the solution of the system (1.1) in the form

$$\sigma_{ij} = h^{-q} \sum_{s=0}^{S} h^s \sigma_{ij}^{(s)}, \quad u_k = h^{-q} \sum_{s=0}^{S} h^s u_k^{(s)}, \quad W = h^{-q} \sum_{s=0}^{S} h^s W^{(s)} \tag{2.1}$$

$$(i = \alpha, \beta, \gamma;\ j = \alpha, \beta, \gamma;\ k = \alpha, \beta)$$

(where q are integers, which are different for different stresses and displacements) and let us construct the iteration processes for the successive determination of coefficients of the above expansions.

The first of these iteration processes consists in the following: a substitution of variables is made in (1.1)

$$\gamma = h\zeta \tag{2.2}$$

the expansions (2.1) are then introduced into the obtained system, in which q is selected as follows:

for the symmetric problem

$$q = 2 \quad \text{for } \sigma_{\alpha\alpha}, \sigma_{\alpha\beta}, \sigma_{\beta\beta}, \qquad q = 1 \quad \text{for } \sigma_{\alpha\gamma}, \sigma_{\beta\gamma}$$
$$q = 0 \quad \text{for } \sigma_{\gamma\gamma}, \qquad q = 2 \quad \text{for } u_\alpha, u_\beta, \qquad q = 1 \quad \text{for } W \tag{2.3}$$

for the skew-symmetric problem

$$q = 2 \quad \text{for } \sigma_{\alpha\alpha}, \sigma_{\alpha\beta}, \sigma_{\beta\beta}, \qquad q = 1 \quad \text{for } \sigma_{\alpha\gamma}, \sigma_{\beta\gamma}$$
$$q = 0 \quad \text{for } \sigma_{\gamma\gamma}, \qquad q = 2 \quad \text{for } u_\alpha, u_\beta, \qquad q = 3 \quad \text{for } W \tag{2.4}$$

and in each equation thus obtained the coefficients of all powers of h are set equal to zero, starting with the lowest. This leads to the following sequences of systems of equations for the coefficients of the expansion (2.1)

$$H_\alpha \frac{\partial \sigma_{\alpha\alpha}^{(s)}}{\partial \alpha} + H_\beta \frac{\partial \sigma_{\alpha\beta}^{(s)}}{\partial \beta} + \frac{\partial \sigma_{\alpha\gamma}^{(s)}}{\partial \zeta} - H_\alpha \frac{\partial \ln H_\beta}{\partial \alpha} (\sigma_{\alpha\alpha}^{(s)} - \sigma_{\beta\beta}^{(s)}) - 2H_\beta \frac{\partial \ln H_\alpha}{\partial \beta} \sigma_{\alpha\beta}^{(s)} = 0 \quad (\alpha\beta)$$

$$H_\alpha \frac{\partial \sigma_{\alpha\gamma}^{(s)}}{\partial \alpha} + H_\beta \frac{\partial \sigma_{\beta\gamma}^{(s)}}{\partial \beta} + \frac{\partial \sigma_{\gamma\gamma}^{(s)}}{\partial \zeta} - H_\alpha \frac{\partial \ln H_\beta}{\partial \alpha} \sigma_{\alpha\gamma}^{(s)} - H_\beta \frac{\partial \ln H_\alpha}{\partial \beta} \sigma_{\beta\gamma}^{(s)} = 0 \tag{2.5}$$

$$E\left(H_\alpha \frac{\partial u_\alpha^{(s)}}{\partial \alpha} - H_\beta \frac{\partial \ln H_\alpha}{\partial \beta} u_\beta^{(s)}\right) = \sigma_{\alpha\alpha}^{(s)} - \nu(\sigma_{\beta\beta}^{(s)} + \sigma_{\gamma\gamma}^{(s-2)}) \quad (\alpha\beta)$$

$$E \frac{\partial W^{(s)}}{\partial \zeta} = \sigma_{\gamma\gamma}^{(a)} - \nu(\sigma_{\alpha\alpha}^{(b)} + \sigma_{\beta\beta}^{(b)})$$

$$E\left(H_\alpha \frac{\partial u_\beta^{(s)}}{\partial \alpha} + H_\beta \frac{\partial u_\alpha^{(s)}}{\partial \beta} + H_\alpha \frac{\partial \ln H_\beta}{\partial \alpha} u_\beta^{(s)} + H_\beta \frac{\partial \ln H_\alpha}{\partial \beta} u_\alpha^{(s)}\right) = 2(1+\nu)\sigma_{\alpha\beta}^{(s)}$$

$$E\left(H_\alpha \frac{\partial W^{(c)}}{\partial \alpha} + \frac{\partial u_\alpha^{(s)}}{\partial \zeta}\right) = 2(1+\nu)\sigma_{\alpha\gamma}^{(s-2)} \quad (\alpha\beta)$$

Here s is the number of a term in the expansion, and the numbers a, b and c are related to s as follows:

$$a = s - 2, \quad b = s, \quad c = s - 2 \text{ for the symmetric problem} \tag{2.6}$$

$$a = s - 4, \quad b = s - 2, \quad c = s \text{ for the skew-symmetric problem} \tag{2.7}$$

In (2.5) and everywhere in what follows the quantities with negative indices are assumed to be equal to zero.

In Equations (2.5), (2.6) and (2.7) the integration with respect to the variable ζ is easily carried out. The result obtained can be written down as: for the symmetric problem

$$W^{(s)} = \zeta w^{(s)} + W^{*(s)}, \quad u_\alpha^{(s)} = v_\alpha^{(s)} + u_\alpha^{*(s)} \quad (\alpha\beta)$$
$$\sigma_{\alpha\alpha}^{(s)} = \tau_{\alpha\alpha}^{(s)} + \sigma_{\alpha\alpha}^{*(s)} \quad (\alpha\beta), \qquad \sigma_{\alpha\beta}^{(s)} = \tau_{\alpha\beta}^{(s)} + \sigma_{\alpha\beta}^{*(s)} \tag{2.8}$$
$$\sigma_{\alpha\gamma}^{(s)} = \zeta\tau_{\alpha\gamma}^{(s)} + \sigma_{\alpha\gamma}^{*(s)} \quad (\alpha\beta), \qquad \sigma_{\gamma\gamma}^{(s)} = S_{\gamma\gamma}^{(s)} + \tfrac{1}{2}\zeta^2\tau_{\gamma\gamma}^{(s)} + \sigma_{\gamma\gamma}^{*(s)}$$

for the skew-symmetric problem

$$W^{(s)} = w^{(s)} + W^{*(s)}, \qquad u_\alpha^{(s)} = \zeta v_\alpha^{(s)} + u_\alpha^{*(s)} \quad (\alpha\beta)$$

$$\sigma_{\alpha\alpha}^{(s)} = \zeta\tau_{\alpha\alpha}^{(s)} + \sigma_{\alpha\alpha}^{*(s)} \quad (\alpha\beta), \qquad \sigma_{\alpha\beta}^{(s)} = \zeta\tau_{\alpha\beta}^{(s)} + \sigma_{\alpha\beta}^{*(s)} \tag{2.9}$$

$$\sigma_{\alpha\gamma}^{(s)} = S_{\alpha\gamma}^{(s)} + \zeta^2\tau_{\alpha\gamma}^{(s)} + \sigma_{\alpha\gamma}^{*(s)} \; (\alpha\beta), \qquad \sigma_{\gamma\gamma}^{(s)} = \zeta S_{\gamma\gamma}^{(s)} + \zeta^3\tau_{\gamma\gamma}^{(s)} + \sigma_{\gamma\gamma}^{*(s)}$$

In these formulas the functions of α and β

$$w^{(s)}, \; v_\alpha^{(s)}, \; v_\beta^{(s)}, \; \tau_{\alpha\alpha}^{(s)}, \; \tau_{\alpha\beta}^{(s)}, \; \tau_{\beta\beta}^{(s)}. \; \tau_{\alpha\gamma}^{(s)}, \; \tau_{\beta\gamma}^{(s)}, \tau_{\gamma\gamma}^{(s)}, \; S_{\alpha\gamma}^{(s)}, \; S_{\beta\gamma}^{(s)}, \; S_{\gamma\gamma}^{(s)}$$

are connected by differential relations which have the form

for the symmetric problem

$$\tau_{\alpha\alpha}^{(s)} = \frac{E}{1-\nu^2} T_{\alpha\alpha}^{(s)} \quad (\alpha\beta), \qquad \tau_{\alpha\beta}^{(s)} = \frac{E}{2(1+\nu)} T_{\alpha\beta}^{(s)}, \qquad \tau_{\alpha\gamma}^{(s)} = T_{\alpha\gamma}^{(s)} \quad (\alpha\beta) \tag{2.10}$$

$$\tau_{\gamma\gamma}^{(s)} = T_{\gamma\gamma}^{(s)}, \qquad Ew^{(s)} = -\nu\left(\tau_{\alpha\alpha}^{(s)} + \tau_{\beta\beta}^{(s)}\right)$$

for the skew-symmetric problem

$$\tau_{\alpha\alpha}^{(s)} = \frac{E}{1-\nu^2} T_{\alpha\alpha}^{(s)} \quad (\alpha\beta), \qquad \tau_{\alpha\beta}^{(s)} = \frac{E}{2(1+\nu)} T_{\alpha\beta}^{(s)}, \qquad \tau_{\alpha\gamma}^{(s)} = \frac{1}{2} T_{\alpha\gamma} \quad (\alpha\beta)$$

$$\tau_{\gamma\gamma}^{(s)} = \frac{1}{3} T_{\gamma\gamma}^{(s)}, \qquad v_\alpha^{(s)} = -H_\alpha \frac{\partial w^{(s)}}{\partial\alpha} \quad (\alpha\beta) \tag{2.11}$$

$$S_{\gamma\gamma}^{(s)} = -\left[H_\alpha \frac{\partial S_{\alpha\gamma}^{(s)}}{\partial\alpha} + H_\beta \frac{\partial S_{\beta\gamma}^{(s)}}{\partial\beta} - H_\alpha \frac{\partial \ln H_\beta}{\partial\alpha} S_{\alpha\gamma}^{(s)} - H_\beta \frac{\partial \ln H_\beta}{\partial\beta} S_{\beta\gamma}^{(s)}\right]$$

In Formulas (2.10) and (2.11)

$$T_{\alpha\alpha}^{(s)} = H_\alpha \frac{\partial v_\alpha^{(s)}}{\partial\alpha} - H_\beta \frac{\partial \ln H_\alpha}{\partial\beta} v_\beta^{(s)} + \nu H_\beta \frac{\partial v_\beta^{(s)}}{\partial\beta} - \nu H_\alpha \frac{\partial \ln H_\beta}{\partial\alpha} v_\alpha^{(s)} \quad (\alpha\beta)$$

$$T_{\alpha\beta}^{(s)} = H_\alpha \frac{\partial v_\beta^{(s)}}{\partial\alpha} + H_\beta \frac{\partial v_\alpha^{(s)}}{\partial\beta} + H_\alpha \frac{\partial \ln H_\beta}{\partial\alpha} v_\beta^{(s)} + H_\beta \frac{\partial \ln H_\alpha}{\partial\beta} v_\alpha^{(s)} \tag{2.12}$$

$$T_{\alpha\gamma}^{(s)} = -H_\alpha \frac{\partial\tau_{\alpha\alpha}^{(s)}}{\partial\alpha} - H_\beta \frac{\partial\tau_{\alpha\beta}^{(s)}}{\partial\beta} + H_\alpha \frac{\partial \ln H_\beta}{\partial\alpha}\left(\tau_{\alpha\alpha}^{(s)} - \tau_{\beta\beta}^{(s)}\right) + 2H_\beta \frac{\partial \ln H_\alpha}{\partial\beta} \tau_{\alpha\beta}^{(s)} \quad (\alpha\beta)$$

$$T_{\gamma\gamma}^{(s)} = -H_\alpha \frac{\partial\tau_{\alpha\gamma}^{(s)}}{\partial\alpha} - H_\beta \frac{\partial\tau_{\beta\gamma}^{(s)}}{\partial\beta} + H_\alpha \frac{\partial \ln H_\beta}{\partial\alpha} \tau_{\alpha\gamma}^{(s)} + H_\beta \frac{\partial \ln H_\alpha}{\partial\beta} \tau_{\beta\gamma}^{(s)}$$

The quantities marked by asterisks in (2.8) and (2.9) are defined by the following relations:

for the symmetric problem

$$Eu_\alpha^{*(s)} = -\int_0^\zeta \left[EH_\alpha \frac{\partial W^{(s-2)}}{\partial\alpha} - 2(1+\nu)\,\sigma_{\alpha\gamma}^{(s-2)}\right] d\zeta \quad (\alpha\beta) \tag{2.13}$$

$$\sigma_{\alpha\alpha}^{*(s)} = \frac{E}{1-\nu^2}\left[H_\alpha \frac{\partial u_\alpha^{*(s)}}{\partial\alpha} - H_\beta \frac{\partial \ln H_\alpha}{\partial\beta} u_\beta^{*(s)} + \nu H_\beta \frac{\partial u_\beta^{*(s)}}{\partial\beta} - \nu H_\alpha \frac{\partial \ln H_\beta}{\partial\alpha} u_\alpha^{*(s)}\right] \quad (\alpha\beta)$$

2.13 (cont.)

$$\sigma_{\alpha\beta}^{*(s)} = \frac{E}{2(1+\nu)}\left[H_\alpha \frac{\partial u_\beta^{*(s)}}{\partial \alpha} + H_\beta \frac{\partial u_\alpha^{*(s)}}{\partial \beta} + H_\alpha \frac{\partial \ln H_\beta}{\partial \alpha} u_\beta^{*(s)} + H_\beta \frac{\partial \ln H_\alpha}{\partial \beta} u_\alpha^{*(s)}\right]$$

$$\sigma_{\alpha\gamma}^{*(s)} = -\int_0^\zeta \left[H_\alpha \frac{\partial \sigma_{\alpha\alpha}^{*(s)}}{\partial \alpha} + H_\beta \frac{\partial \sigma_{\alpha\beta}^{*(s)}}{\partial \beta} - H_\alpha \frac{\partial \ln H_\beta}{\partial \alpha}(\sigma_{\alpha\alpha}^{*(s)} - \sigma_{\beta\beta}^{*(s)}) - \right.$$

$$\left. - 2H_\beta \frac{\partial \ln H_\alpha}{\partial \beta} \sigma_{\alpha\beta}^{*(s)}\right] d\zeta \qquad (\alpha\beta)$$

$$\sigma_{\gamma\gamma}^{*(s)} = -\int_0^\zeta \left[H_\alpha \frac{\partial \sigma_{\alpha\gamma}^{*(s)}}{\partial \alpha} + H_\beta \frac{\partial \sigma_{\beta\gamma}^{*(s)}}{\partial \beta} - H_\alpha \frac{\partial \ln H_\beta}{\partial \alpha} \sigma_{\alpha\gamma}^{*(s)} - H_\beta \frac{\partial \ln H_\alpha}{\partial \beta} \sigma_{\beta\gamma}^{*(s)}\right] d\zeta$$

$$EW^{*(s)} = \int_0^\zeta [\sigma_{\gamma\gamma}^{(s-2)} - \nu(\sigma_{\alpha\alpha}^{*(s)} + \sigma_{\beta\beta}^{*(s)})]\, d\zeta$$

for the skew-symmetric problem

$$EW^{*(s)} = +\int_0^\zeta [\sigma_{\gamma\gamma}^{(s-4)} - \nu(\sigma_{\alpha\alpha}^{(s-2)} + \sigma_{\beta\beta}^{(s-2)})]\, d\zeta \qquad (2.14)$$

$$Eu_\alpha^{*(s)} = -E\int_0^\zeta H_\alpha \frac{\partial W^{*(s)}}{\partial \alpha} d\zeta + 2(1+\nu)\int_0^\zeta \sigma_{\alpha\gamma}^{(s-2)} d\zeta \qquad (\alpha\beta)$$

$$\sigma_{\alpha\alpha}^{*(s)} = \frac{E}{1-\nu^2}\left[H_\alpha \frac{\partial u_\alpha^{*(s)}}{\partial \alpha} - H_\beta \frac{\partial \ln H_\alpha}{\partial \beta} u_\beta^{*(s)} + \nu H_\beta \frac{\partial u_\beta^{*(s)}}{\partial \beta} - \right.$$

$$\left. - \nu H_\alpha \frac{\partial \ln H_\beta}{\partial \alpha} u_\alpha^{*(s)}\right] + \frac{\nu}{1-\nu} \sigma_{\gamma\gamma}^{(s-2)} \qquad (\alpha\beta)$$

. . . .

(the expressions for $\sigma_{\alpha\beta}^{*(s)}$, $\sigma_{\alpha\gamma}^{*(s)}$, $\sigma_{\gamma\gamma}^{*(s)}$ are not written down; they are the same for the skew-symmetric problem as for the symmetric).

Formulas (2.13) and (2.14) are of recurrent character and allow the determination of quantities with asterisks associated with the approximation (s), without solving any equations, if all the quantities associated with the approximations (0), (1), (2), ... $(s-1)$ are known.

3. We will assume that the functions $a, b, c_\alpha, c_\beta, d_\alpha, d_\beta$ in (1.2) are independent of h. (A generalization for the case in which these quantities are polynomials of integral or fractional powers of h is made in an obvious manner, with the help of the superposition principle). With this assumption, the boundary conditions (1.3) and (1.5) imposed on stresses can be replaced by sequences of conditions imposed on the coefficients of the expansions (2.1)

for the symmetric case

$$\sigma_{\gamma\gamma}^{(0)} = \tfrac{1}{2} q, \qquad \sigma_{\alpha\gamma}^{(0)} = \pm\tfrac{1}{2} q_\alpha, \qquad \sigma_{\beta\gamma}^{(0)} = \pm\tfrac{1}{2} q_\beta \qquad \text{for } \zeta = \pm 1 \qquad (3.1)$$

$$\sigma_{\gamma\gamma}^{(t)} = \sigma_{\alpha\gamma}^{(t)} = \sigma_{\beta\gamma}^{(t)} = 0 \qquad (t > 0)$$

for the skew-symmetric case

$$\sigma_{\gamma\gamma}^{(0)} = +\frac{1}{2}p, \quad \sigma_{\alpha\gamma}^{(0)} = \frac{1}{2}p_\alpha, \quad \sigma_{\beta\gamma}^{(0)} = \frac{1}{2}p_\beta \quad \text{for } \zeta = \pm 1$$
$$\sigma_{\gamma\gamma}^{(t)} = \sigma_{\alpha\gamma}^{(t)} = \sigma_{\beta\gamma}^{(t)} = 0 \qquad (t > 0) \tag{3.2}$$

Substituting (2.8) and (2.9) into (3.1) and (3.2) and bearing in mind that for $s = 0$ and $s = 1$ the quantities marked by asterisks are equal to zero, which follows from (2.13) and (2.14), we obtain

for the symmetric problem

$$\tau_{\alpha\gamma}^{(s)} = \frac{1}{2}q_\alpha^{(s)} \quad (\alpha\beta), \qquad S_{\gamma\gamma}^{(s)} + \frac{1}{2}\tau_{\gamma\gamma}^{(s)} = \frac{1}{2}q^{(s)} \tag{3.3}$$

where

$$q^{(0)} = q, \quad q^{(1)} = 0, \quad q^{(r)} = -2\sigma_{\gamma\gamma}^{*(r)}|_{\zeta=1} \quad (r>1)$$
$$q_\alpha^{(0)} = q_\alpha, \quad q_\alpha^{(1)} = 0, \quad q_\alpha^{(r)} = -2\sigma_{\alpha\gamma}^{*(r)}|_{\zeta=1} \quad (\alpha\beta) \quad (r>1) \tag{3.4}$$

for the skew-symmertic problem

$$S_{\alpha\gamma}^{(s)} + \tau_{\alpha\gamma}^{(s)} = {}^1/_2 p_\alpha^{(s)} \quad (\alpha\beta), \qquad S_{\gamma\gamma}^{(s)} + \tau_{\gamma\gamma}^{(s)} = {}^1/_2 p^{(s)} \tag{3.5}$$

where

$$p^{(0)} = p, \quad p^{(1)} = 0, \quad p^{(r)} = -2\sigma_{\gamma\gamma}^{*(r)}|_{\zeta=1}$$
$$p_\alpha^{(0)} = p_\alpha, \quad p_\alpha^{(1)} = 0, \quad p_\alpha^{(r)} = -2\sigma_{\alpha\gamma}^{*(r)}|_{\zeta=1} \quad (\alpha\beta) \quad (r>1) \tag{3.6}$$

Equalities (2.10), (2.12) and (3.3) from a system of ten equations with the unknowns

$$v_\alpha^{(s)},\ v_\beta^{(s)},\ w^{(s)},\ \tau_{\alpha\alpha}^{(s)},\ \tau_{\alpha\beta}^{(s)},\ \tau_{\beta\beta}^{(s)},\ \tau_{\alpha\gamma}^{(s)},\ \tau_{\beta\gamma}^{(s)},\ \tau_{\gamma\gamma}^{(s)},\ S_{\gamma\gamma}^{(s)} \tag{3.7}$$

By means of the last two equalities (2.10), (2.12) and equalities (3.3) and (3.4) the quantities

$$\tau_{\alpha\gamma}^{(s)},\quad \tau_{\beta\gamma}^{(s)},\quad \tau_{\gamma\gamma}^{(s)},\quad S_{\gamma\gamma}^{(s)} \tag{3.8}$$

are expressed in terms of

$$q,\ q_\alpha,\ q_\beta,\ \sigma_{\gamma\gamma}^{*(s)}|_{\zeta=1},\ \sigma_{\alpha\gamma}^{*(s)}|_{\zeta=1},\ \sigma_{\beta\gamma}^{*(s)}|_{\zeta=1}$$

and the quantity $w^{(s)}$ is expressed in terms of $\tau_{\alpha\alpha}^{(s)}$ and $\tau_{\beta\beta}^{(s)}$ or in terms of $v_\alpha^{(s)}$ and $v_\beta^{(s)}$ which follows from (2.10), (2.12).

Substituting values thus obtained (3.8) into the first three equalities (2.10) and (2.12), we arrive at a system of five equations with the unknowns $v_\alpha^{(s)}$, $v_\beta^{(s)}$, $\tau_{\alpha\alpha}^{(s)}$, $\tau_{\alpha\beta}^{(s)}$, $\tau_{\beta\beta}^{(s)}$

$$H_\alpha \frac{\partial \tau_{\alpha\alpha}^{(s)}}{\partial\alpha} + H_\beta \frac{\partial \tau_{\alpha\beta}^{(s)}}{\partial\beta} - H_\alpha \frac{\partial \ln H_\beta}{\partial\alpha}(\tau_{\alpha\alpha}^{(s)} - \tau_{\beta\beta}^{(s)}) - 2H_\beta \frac{\partial \ln H_\alpha}{\partial\beta}\tau_{\alpha\beta}^{(s)} = -\frac{1}{2}q_\alpha^{(s)} \quad (\alpha\beta)$$

$$\tau_{\alpha\alpha}^{(s)} = \frac{E}{1-\nu^2}\left[H_\alpha \frac{\partial v_\alpha^{(s)}}{\partial\alpha} - H_\beta \frac{\partial \ln H_\alpha}{\partial\beta} v_\beta^{(s)} + \nu H_\beta \frac{\partial v_\beta^{(s)}}{\partial\beta} - \nu H_\alpha \frac{\partial \ln H_\beta}{\partial\alpha} v_\alpha^{(s)}\right] \quad (\alpha\beta) \tag{3.9}$$

$$\tau_{\alpha\beta}^{(s)} = \frac{E}{2(1+\nu)}\left[H_\alpha \frac{\partial v_\beta^{(s)}}{\partial\alpha} + H_\beta \frac{\partial v_\alpha^{(s)}}{\partial\beta} + H_\alpha \frac{\partial \ln H_\beta}{\partial\alpha} v_\beta^{(s)} + H_\beta \frac{\partial \ln H_\alpha}{\partial\beta} v_\alpha^{(s)}\right]$$

For each (s) this system is equivalent to the nonhomogeneous system of differential equations of the generalized plane stress problem.

The first equality (3.9) represents the equilibrium equations and the last two are the relations of elasticity. The constant terms on the right-hand sides of the equilibrium equations can be looked upon as components of a certain hypothetical tangential loading. For $s = 0$ the quantities $q_\alpha^{(s)}$ and $q_\beta^{(s)}$ coincide with the components q_α, q_β of the load actually applied; for $s = 1$ $q_\alpha^{(1)} = q_\beta^{(1)} = 0$, and for $s > 1$ these quantities are defined as certain differential operators of q_α, q_β. In particular, it is easily verified that if q_α, q_β have a potential function $V^{(0)}$, then the potential function $V^{(2)}$ for $q_\alpha^{(2)}$, $q_\beta^{(2)}$ is given by Formula

$$V^{(2)} = \frac{4-3\nu}{12(1-\nu)} \Delta V^{(0)} \quad (\Delta \text{ is the Laplace operator}) \tag{3.10}$$

For the skew-symmetric problem Equations (2.11), (2.12) and (3.5) form a system of twelve equations with the unknowns

$$v_\alpha^{(s)},\ v_\beta^{(s)},\ w^{(s)},\ \tau_{\alpha\alpha}^{(s)},\ \tau_{\alpha\beta}^{(s)},\ \tau_{\beta\beta}^{(s)},\ \tau_{\alpha\gamma}^{(s)},\ \tau_{\beta\gamma}^{(s)},\ \tau_{\gamma\gamma}^{(s)},\ S_{\alpha\gamma}^{(s)},\ S_{\beta\gamma}^{(s)},\ S_{\gamma\gamma}^{(s)} \tag{3.11}$$

By means of the last two equations (2.11) and equalities (2.12), (3.5), (3.6) the quantities

$$S_{\alpha\gamma}^{(s)},\quad S_{\beta\gamma}^{(s)},\quad S_{\gamma\gamma}^{(s)},\quad \tau_{\gamma\gamma}^{(s)} \tag{3.12}$$

can be expressed in terms of

$$\tau_{\alpha\gamma}^{(s)}, \tau_{\beta\gamma}^{(s)}, \tau_{\gamma\gamma}^{(s)}, p, p_\alpha, p_\beta, \sigma_{\alpha\gamma}^{*(s)}\big|_{\zeta=1}, \sigma_{\beta\gamma}^{*(s)}\big|_{\zeta=1}, \sigma_{\gamma\gamma}^{*(s)}\big|_{\zeta=1},$$

and $v_\alpha^{(s)}$ and $v_\beta^{(s)}$ in terms of $w^{(s)}$.

Substituting these results into the first four equalities (2.11) we obtain a system of six equations with the unknowns

$$\tau_{\alpha\alpha}^{(s)},\ \tau_{\alpha\beta}^{(s)},\ \tau_{\beta\beta}^{(s)},\ \tau_{\alpha\gamma}^{(s)},\ \tau_{\beta\gamma}^{(s)},\ w^{(s)} \tag{3.13}$$

It has the form

$$\tau_{\alpha\alpha}^{(s)} = -\frac{E}{1-\nu^2}\left[H_\alpha \frac{\partial}{\partial\alpha}\left(H_\alpha \frac{\partial w^{(s)}}{\partial\alpha}\right) - H_\beta \frac{\partial \ln H_\alpha}{\partial\beta} H_\beta \frac{\partial w^{(s)}}{\partial\beta} + \right.$$
$$\left. + \nu H_\beta \frac{\partial}{\partial\beta}\left(H_\beta \frac{\partial w^{(s)}}{\partial\beta}\right) - \nu H_\alpha \frac{\partial \ln H_\beta}{\partial\alpha} H_\alpha \frac{\partial w^{(s)}}{\partial\alpha}\right] \quad (\alpha\beta) \tag{3.14}$$

$$\tau_{\alpha\beta}^{(s)} = -\frac{E}{2(1+\nu)}\left[H_\alpha \frac{\partial}{\partial\alpha}\left(H_\beta \frac{\partial w^{(s)}}{\partial\beta}\right) + H_\beta \frac{\partial}{\partial\beta}\left(H_\alpha \frac{\partial w^{(s)}}{\partial\alpha}\right) + \right.$$
$$\left. + H_\alpha \frac{\partial \ln H_\beta}{\partial\alpha} H_\beta \frac{\partial w^{(s)}}{\partial\beta} + H_\beta \frac{\partial \ln H_\alpha}{\partial\beta} H_\alpha \frac{\partial w^{(s)}}{\partial\alpha}\right]$$

$$H_\alpha \frac{\partial \tau_{\alpha\alpha}^{(s)}}{\partial\alpha} + H_\beta \frac{\partial \tau_{\alpha\beta}^{(s)}}{\partial\beta} - H_\alpha \frac{\partial \ln H_\beta}{\partial\alpha}\left(\tau_{\alpha\alpha}^{(s)} - \tau_{\beta\beta}^{(s)}\right) - 2H_\beta \frac{\partial \ln H_\alpha}{\partial\beta}\tau_{\alpha\beta}^{(s)} = -2\tau_{\alpha\gamma}^{(s)} \quad (\alpha\beta)$$

$$H_\alpha \frac{\partial \tau_{\alpha\gamma}^{(s)}}{\partial\alpha} + H_\beta \frac{\partial \tau_{\beta\gamma}^{(s)}}{\partial\beta} - H_\alpha \frac{\partial \ln H_\beta}{\partial\alpha}\tau_{\alpha\gamma}^{(s)} - H_\beta \frac{\partial \ln H_\alpha}{\partial\beta}\tau_{\beta\gamma}^{(s)} =$$
$$= \frac{3}{4}\left\{p^{(s)} + H_\alpha H_\beta\left[\frac{\partial}{\partial\alpha}\left(\frac{p_\alpha^{(s)}}{H_\beta}\right) + \frac{\partial}{\partial\beta}\left(\frac{p_\beta^{(s)}}{H_\alpha}\right)\right]\right\}$$

This system is equivalent to the system of differential equations of the classical theory of plate bending. Namely, if the following new quantities are introduced by means of Formulas

$$M_{\alpha}^{(s)} = \frac{2}{3} h^s \tau_{\alpha\alpha}^{(s)} + h^s \int_{-1}^{+1} \zeta \sigma_{\alpha\alpha}^{*\,(s)} d\zeta \qquad (\alpha\beta)$$

$$H_{\alpha\beta}^{(s)} = \frac{2}{3} h^s \tau_{\alpha\beta}^{(s)} + h^s \int_{-1}^{+1} \zeta \sigma_{\alpha\beta}^{*\,(s)} d\zeta \qquad (3.15)$$

$$N_{\alpha}^{(s)} = h^s \left[-\frac{4}{3} \tau_{\alpha\gamma}^{(s)} + p_{\alpha}^{(s)} + \int_{-1}^{+1} \sigma_{\alpha\gamma}^{*\,(s)} d\zeta \right] \qquad (\alpha\beta)$$

and the first five quantities in (3.13) are expressed in terms of $M_{\alpha}^{(s)}$, $M_{\beta}^{(s)}$, $H_{\alpha\beta}^{(s)}$, $N_{\alpha}^{(s)}$, $N_{\beta}^{(s)}$ and substituted into equations (3.14), then we obtain the usual relations of the plate bending theory in which the quantities (3.15) take the role of moments and shearing forces in the sth approximation (*)

It has been shown in [2] (for the cylindrical coordinate system), that all of these quantities can be expressed in terms of the function $w_0^{(s)} = h^{s-3} w^{(s)}$, which satisfies Equation

$$\Delta\Delta w_0^{(s)} = \frac{P^{(s)}}{D} \qquad \left(D = \frac{2Eh^3}{3(1-\nu^2)} \right) \qquad (3.16)$$

Here $P^{(s)}$ is expressed in terms of p, p_{α}, p_{β}. In particular

$$P^{(0)} = p + H_{\alpha} H_{\beta} \left[\frac{\partial}{\partial\alpha} \left(\frac{p_{\alpha}}{H_{\beta}} \right) + \frac{\partial}{\partial\beta} \left(\frac{p_{\beta}}{H_{\alpha}} \right) \right], \qquad P^{(1)} = 0 \qquad (3.17)$$

$$P^{(2)} = \frac{h^2}{30(1-\nu^2)} \left\{ -3(8-3\nu)\Delta p + (4+\nu)\Delta \left[H_{\alpha} H_{\beta} \frac{\partial}{\partial\alpha} \left(\frac{p_{\alpha}}{H_{\beta}} \right) + H_{\alpha} H_{\beta} \frac{\partial}{\partial\beta} \left(\frac{p_{\beta}}{H_{\alpha}} \right) \right] \right\}$$

For $p_{\alpha} = p_{\beta} = 0$ we have

$$P^{(0)} = p, \qquad P^{(1)} = 0, \qquad P^{(2)} = -\frac{(8-3\nu)h^2}{10(1-\nu)} \Delta p \qquad (3.18)$$

Note. It can be seen from Formulas (3.10), (3.17), (3.18), (3.4), (3.6), (2.13) and (2.14) that in the construction of an approximate theory of bending and extension of a plate it is required that the surface loads have continuous derivatives of sufficiently high order. It should be pointed out that the requirement of differentiability of load is not a shortcoming of the method under consideration, but a reflection of the physical nature of the problem. This can easily be verified with the example of bending of a circular plate referred to polar coordinates and acted upon by a normal load of intensity $p = p_0 \cos n\theta$

At the center of the plate the function p becomes nondifferentiable and the method discussed here cannot be applied. This is explained by the fact that at the plate center in this case the state of stress is essentially three-dimensional and no theory based on the assumption of small plate thickness is capable of describing it.

4. In [1] the iteration process presented above was termed basic. It is designed for the construction of the basic states of stress, deeply penetrating inside the plate. Therefore, in the formulation of Equations (2.5) it was assumed that the stresses and displacements do not vary too rapidly along α, β, and the fact that in a thin plate the stresses and displacements must vary rapidly with γ was taken into consideration by means of the substitution of variables (2.2).

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*) In [1], for the problem of bending of a plate, the substitution (3.15) was carried out in Cartesian coordinates, and quantities with asterisks were erroneously left out of account.

Formulas (2.8), (2.9), (2.13), (2.14) show that the basic iteration process determines such states of stress of the plate in which the stresses and displacements with respect to the variable ζ , i.e. along the thickness, vary according to a polynomial law, the degree of the polynomials increasing unboundedly with the increase of the number of approximations. As the number of approximations increases without limits, the states of stress will be obtained, in which stresses and displacements are expressed by power series in ζ . Therefore, using a sufficient number of approximations and taking advantage of the arbitrary constants of integration of equations (3.9) and (3.14), one can formally satisfy the boundary conditions on the side surfaces of the plate with any degree of accuracy.

This constitutes one of the possible ways of constructing an approximate theory of plates; it represents one of the variants of the power series method, i.e. a method which has been used repeatedly (see, for instance, [3 to 6], and which is based on the expansion of the unknown quantities into power series in the direction of the plate thickness. This method can be applied for any thickness of the plate (including the large one), although it should be noted that the difficulties associated with the investigation of the character of convergence of corresponding series have do far not been overcome. However, if one speaks of thin plates only, the power series method has a drawback the nature of which will become apparent below.

From the above discussion it follows that for the equations of the theory of elasticity the process of constructing integrals which are expressed in terms of power series in one of the variables can be arranged in such a manner that it will have iterational character, and, at all stages, quantities of the same order of magnitude (and independent of h) will be taken into account. This offers obvious advantages, especially if one bears in mind that at each stage the well known equations of the problems of plate bending and generalized plane stress have to be integrated. However, all these advantages are almost completely lost when the boundary conditions on the side surfaces of the plate have to be satisfied: to carry that operation out one has to construct a certain number of approximations in a general form, write down the corresponding expressions for stresses or displacements, and state the requirement that those quantities should, in one sense or another, approximately satisfy the boundary conditions. Thus, in the variant of the power series method discussed here the arbitrary constants of integration have to be determined at once, and not separately for each approximation, and besides, the computations will now contain quantities proportional to various powers of h . In other variants of the power series method its shortcomings remain essentially the same and are manifested by the fact that with the increase of the number of approximations the order of the corresponding equations increases, the coefficients of the equations depend on h , and for small values of h. are substantially different from each other in absolute value.

The property of the power series method described above is not accidental and cannot be explained by an unfortunate arrangement of the calculations. In the classical plate theory only such characteristics of edge force action as tractions and moments are taken into consideration for the fulfillment of boundary conditions on the side surfaces. The improved accuracy in satisfying the boundary conditions is equivalent to taking into account the self-equilibrated edge influences (polymoments), and this, according to St.Venant's principle, leads to the appearance of rapidly damped states of stress near the edges.

If the equations of an approximate plate theory are found with sufficient accuracy and have the purpose of describing all the elastic phenomena, they must contain the corresponding rapidly damped integrals which are known in the theory of asymptotic integration as the boundary layer [7 and 8]. Such integrals are precisely the ones which have equations obtained by the power series method, i.e. equations of sufficiently high order whose coefficients contain a small parameter.

The above considerations naturally lead to the idea of devoting a separate investigation to rapidly damped states of stress and working out for them such iteration processes which are better adapted for the purpose than the basic iteration process. The method in which this idea is developed was considered in [1] and [9 to 15] and can be called asymptotic. In the present study an auxiliary iteration process is formulated for the determination of rapidly damped states of stress in a plate subjected to bending and extension, and it is shown that if the state of stress in a plate is sought as a sum of states of stress which are obtained by means of the basic and the auxiliary processes, then not only the process of integration of equations, but also the process of imposing the boundary conditions on the side surfaces will have the iterational character.

Lately some papers have appeared [16 and 17] in which the approximate bending theory of plates is constructed by means of the symbolic method of Lur'e [18]. In this method equations of infinitely high order are obtained, and in that sense it borders on the power series method. It has been shown in [16] that the results arrived at by this method are quite close to those obtained by the asymptotic method.

N o t e . The differential equations which arise in the power series method can be solved by asymptotic integration as equations containing a small parameter. In the process, the rapidly damped solutions are automatically singled out, and the difference between the asymptotic method and that of the power series becomes inessential. Then the equations arising in the power series method loose their independent significance and turn into an auxiliary means for obtaining results to which the asymptotic method leads directly.

5. Let us now describe the auxiliary iteration process, assuming that it is to be used in constructing states of stress rapidly damped in the direction away from the line

$$\alpha = \alpha_0$$

which is supposed to coincide with the edge of the plate.

This process consists in the following: in (1.1) we make a substitution of variables

$$H_\alpha \frac{\partial}{\partial \alpha} = h^{-1} \frac{\partial}{\partial \xi}, \qquad \frac{\partial}{\partial \gamma} = h^{-1} \frac{\partial}{\partial \zeta} \tag{5.1}$$

and introduce the expansions (2.1) into the transformed system, choosing q such that

$$q = r \quad \text{for } \sigma_{\alpha\alpha}, \sigma_{\alpha\beta}, \sigma_{\beta\beta}, \sigma_{\alpha\gamma}, \sigma_{\beta\gamma}, \sigma_{\gamma\gamma}$$
$$q = r - 1 \quad \text{for } u_\alpha, u_\beta, W \tag{5.2}$$

(r is an undetermined number, so far), then we set the coefficients at all powers of h equal to zero, starting with the lowest one. This leads to the following sequence of equations for the coefficients of expansion (2.1):

$$\frac{\partial \sigma_{\alpha\beta}^{(s)}}{\partial \xi} + \frac{\partial \sigma_{\beta\gamma}^{(s)}}{\partial \zeta} = R_\beta^{(s-1)}$$
$$E \frac{\partial u_\beta^{(s)}}{\partial \xi} - 2(1+\nu)\sigma_{\alpha\beta}^{(s)} = -EH_\beta \frac{\partial u_\alpha^{(s-1)}}{\partial \beta} + Ek_\beta u_\beta^{(s-1)} \tag{5.3}$$
$$E \frac{\partial u_\beta^{(s)}}{\partial \zeta} - 2(1+\nu)\sigma_{\beta\gamma}^{(s)} = -EH_\beta \frac{\partial W^{(s-1)}}{\partial \beta}$$

$$\frac{\partial \sigma_{\alpha\alpha}^{(s)}}{\partial \xi} + \frac{\partial \sigma_{\alpha\gamma}^{(s)}}{\partial \zeta} = R_\xi^{(s-1)}, \qquad \frac{\partial \sigma_{\alpha\gamma}^{(s)}}{\partial \xi} + \frac{\partial \sigma_{\gamma\gamma}^{(s)}}{\partial \zeta} = R_\zeta^{(s-1)}$$
$$E \frac{\partial u_\alpha^{(s)}}{\partial \xi} = [\sigma_{\alpha\alpha}^{(s)} - \nu(\sigma_{\beta\beta}^{(s)} + \sigma_{\gamma\gamma}^{(s)})], \quad E\frac{\partial W^{(s)}}{\partial \zeta} = [\sigma_{\gamma\gamma}^{(s)} - \nu(\sigma_{\alpha\alpha}^{(s)} + \sigma_{\beta\beta}^{(s)})] \tag{5.4}$$
$$E\left(\frac{\partial W^{(s)}}{\partial \xi} + \frac{\partial u_\alpha^{(s)}}{\partial \zeta}\right) - 2(1+\nu)\sigma_{\alpha\gamma}^{(s)} = 0$$
$$\sigma_{\beta\beta}^{(s)} - \nu(\sigma_{\alpha\alpha}^{(s)} + \sigma_{\gamma\gamma}^{(s)}) = EH_\beta \frac{\partial u_\beta^{(s-1)}}{\partial \beta} + Ek_\beta u_\alpha^{(s-1)}$$

The designations used in Equations (5.3) and (5.4) are

$$R_\xi^{(s-1)} = -H_\beta \frac{\partial\sigma_{\alpha\beta}^{(s-1)}}{\partial\beta} - k_\beta\left(\sigma_{\alpha\alpha}^{(s-1)} - \sigma_{\beta\beta}^{(s-1)}\right) \tag{5.5}$$

$$R_\zeta^{(s-1)} = H_\beta \frac{\partial\sigma_{\beta\gamma}^{(s-1)}}{\partial\beta} - k_\beta\sigma_{\alpha\gamma}^{(s-1)}, \qquad R_\beta^{(s-1)} = -H_\beta \frac{\partial\sigma_{\beta\beta}^{(s-1)}}{\partial\beta}\ 2k_\beta\sigma_{\alpha\beta}^{(s-1)}$$

also taken into account are formulas for the curvatures of the coordinate lines

$$k_\alpha = H_\beta \frac{\partial \ln H_\alpha}{\partial\beta}, \qquad k_\beta = -H_\alpha \frac{\partial \ln H_\beta}{\partial\alpha}$$

Moreover, it has been assumed that $k_\alpha = 0$. The latter assumption means that, for the purpose of constructing rapidly damped states of stress, the coordinate system α,β,γ in which the basic state of stress was constructed has been replaced by a new coordinate system ξ,β,ζ in which the ξ-lines are straight. The ξ,β,ζ coordinate system should be used only near the edge $\alpha = \alpha_0$.

This is possible, in general, but it should be pointed out that the edge corner points (whether rounded off or not) should be excluded from consideration.

Equations (5.3) and (5.4) from a reccurence system. From it, quantities with the index (s) are successively determined in terms of quantities with the index $(s-1)$. For $s = 0$, when $(s-1)$ quantities vanish, the homogeneous equations well known in the theory of elasticity are obtained. Equalities (5.3) represent the equations of torsion of bars (with respect to β-axis), and equalities (5.4) are the equations of the plane strain problem (in the $\xi\zeta$ plane). For $s > 0$ (when $(s-1)$ quantities should be regarded as known) the equations remain the same, but become nonhomogeneous.

In what follows it will be convenient to represent the coefficients of the expansions (2.1) in the form

$$\sigma_{ij}^{(s)} = \sigma_{ij\,(1)}^{(s)} + \sigma_{ij\,(2)}^{(s)}, \qquad u_k^{(s)} = u_{k\,(1)}^{(s)} + u_{k\,(2)}^{(s)}, \qquad W^{(s)} = W_{(1)}^{(s)} + W_{(2)}^{(s)}$$
$$(i, j = \alpha, \beta, \gamma;\ k = \alpha, \beta)$$

defining the quantities with the additional indices (1) and (2) in the following way:

in the zero approximation

$$\sigma_{\alpha\alpha\,(1)}^{(0)} = \sigma_{\beta\beta\,(1)}^{(0)} = \sigma_{\alpha\gamma\,(1)}^{(0)} = \sigma_{\gamma\gamma\,(1)}^{(0)} = u_{\alpha\,(1)}^{(0)} = W_{(1)}^{(0)} \equiv 0 \tag{5.6}$$

$\sigma_{\alpha\beta\,(1)}^{(0)}$, $\sigma_{\beta\gamma\,(1)}^{(0)}$, $u_{\beta\,(1)}^{(0)}$ are defined from the homogeneous system (5.3)

$$\sigma_{\alpha\beta\,(2)}^{(0)} = \sigma_{\beta\gamma\,(2)}^{(0)} = u_{\beta\,(2)}^{(0)} \equiv 0 \tag{5.7}$$

$\sigma_{\alpha\alpha\,(2)}^{(0)}$, $\sigma_{\beta\beta\,(2)}^{(0)}$, $\sigma_{\alpha\gamma\,(2)}^{(0)}$, $\sigma_{\gamma\gamma\,(2)}^{(0)}$, $u_{\alpha\,(2)}^{(0)}$, $W_{(2)}^{(0)}$ are determined from the homogeneous system (5.4).

In the sth approximation $(s > 0)$ the quantities with additional indices (1) and (2) must satisfy the nonhomogeneous system (5.3), (5.4) in which the quantities on the right-hand sides have indices (1) or (2), respectively.

N o t e . Quantities with additional indices (1) and (2) correspond to those which in [1] were determined by means of the first and second variants of the auxiliary iteration process.

We will require that solutions of the systems (5.3) and (5.4) satiffy the conditions

$$\sigma_{\alpha\gamma}^{(s)} \equiv \sigma_{\alpha\gamma(1)}^{(s)} + \sigma_{\alpha\gamma(2)}^{(s)} = 0 \quad (\alpha\beta) \qquad \text{for } \zeta = \pm 1 \tag{5.8}$$
$$\sigma_{\gamma\gamma}^{(s)} \equiv \sigma_{\gamma\gamma(1)}^{(s)} + \sigma_{\gamma\gamma(2)}^{(s)} = 0$$

Then for the state of stress which represents a sum of the states of stress determined by the basic and the auxiliary iteration processes, conditions (1.3) or (1.5) will be satidfied.

In the formulation of the auxiliary iteration process the substitution of variables (5.1) was used and it was assumed that the variation of stresses and displacements along ξ, β, ζ is not too large. This is equivalent to the assumption that along α and γ stresses and displacements do have large variation. Following the way outlined in Section 4, care should be taken to insure that rapid variability along α would imply rapid damping of the unknown quantities, i.e. conditions of damping should be imposed.

Considering this problem, we assume that the variable ξ is determined from (5.1) by Formula

$$\xi = \frac{1}{h} \int_{\alpha_0}^{\alpha} \frac{d\alpha}{H_\alpha}$$

Then the edge $\alpha = \alpha_0$ will be defined by Equation $\xi = 0$, and Equations (5.3) and (5.4) will have to be integrated in the half-strip

$$0 \geqslant \xi > -\infty, \qquad -1 \leqslant \zeta \leqslant +1 \tag{5.9}$$

As pointed out above, here the systems (5.3) and (5.4) have a definite pgysical meaning: they represent the nonhomogeneous equations of the problems of torsion and plane strain.

Conditions (5.8), i.e. stress-free conditions must be satisfied on the straight lines $\zeta \pm 1$. For the damped solutions when $\xi = -\infty$ the stresses and displacements vanish. Thus, the half-strip (5.9) will be acted upon only by

a) edge forces

$$[\sigma_{\alpha\alpha(1)}^{(s)} + \sigma_{\alpha\alpha(2)}^{(s)}]_{\xi=0}, \qquad [\sigma_{\alpha\gamma(1)}^{(s)} + \sigma_{\alpha\gamma(2)}^{(s)}]_{\xi=0}, \qquad [\sigma_{\alpha\beta(1)}^{(s)} + \sigma_{\alpha\beta(2)}^{(s)}]_{\xi=0} \tag{5.10}$$

distributed along the straight line $\xi = 0$;

b) mass forces whose components R_ξ, R_β, R_ζ are determined from Formulas (5.5).

For $s = 0$ components of forces (5.5) will vanish and, due to St.Venant's principle, conditions of damping will be the requirements of equilibrium of forces (5.10), i.e. if (5.6) and (5.7) are taken into account, this will furnish the equalities

$$\int_{-1}^{+1} \zeta \sigma_{\alpha\alpha(2)}^{(0)} |_{\xi=0} \, d\zeta = 0, \qquad \int_{-1}^{+1} \sigma_{\alpha\gamma(2)}^{(0)} |_{\xi=0} \, d\zeta = 0 \tag{5.11}$$

$$\int_{-1}^{+1} \sigma_{\alpha\alpha(2)}^{(0)} |_{\xi=0} \, d\zeta = 0, \qquad \int_{-1}^{+1} \sigma_{\alpha\beta(1)}^{(0)} |_{\xi=0} \, d\zeta = 0 \tag{5.12}$$

Suppose, furthermore, that there exist $(s - 1)$ approximations, determined from Equations (5.3) and (5.4), which satisfy the conditions of damping.

Then $R_\xi^{(s-1)}$, $R_\zeta^{(s-1)}$, $R_\beta^{(s-1)}$ will not vanish, but the corresponding load will be localized in a narrow zone adjacent to the straight line $\xi = 0$.

The conditions of existence of damped solutions for the sth approximation will be the requirements

$$\int_{-1}^{+1} \zeta\,[\sigma_{\alpha\alpha(1)}^{(s)} + \sigma_{\alpha\alpha(2)}^{(s)}]_{\xi=0}\,d\zeta = \int_{-\infty}^{0} d\xi \int_{-1}^{+1} [\zeta R_\xi{}^{(s-1)} - \xi R_\zeta{}^{(s-1)}]\,d\zeta$$
$$\int_{-1}^{+1} [\sigma_{\alpha\gamma(1)}^{(s)} + \sigma_{\alpha\gamma(2)}^{(s)}]_{\xi=0}\,d\zeta = \int_{-\infty}^{0} d\xi \int_{-1}^{+1} R_\zeta{}^{(s-1)}\,d\zeta \tag{5.13}$$

$$\int_{-1}^{+1} [\sigma_{\alpha\alpha(1)}^{(s)} + \sigma_{\alpha\alpha(2)}^{(s)}]_{\xi=0}\,d\zeta = \int_{-\infty}^{0} d\xi \int_{-1}^{+1} R_\xi{}^{(s-1)}\,d\zeta$$
$$\int_{-1}^{+1} [\sigma_{\alpha\beta(1)}^{(s)} + \sigma_{\alpha\beta(2)}^{(s)}]_{\xi=0}\,d\zeta = \int_{-\infty}^{0} d\xi \int_{-1}^{+1} R_\beta{}^{(s-1)}\,d\zeta \tag{5.14}$$

which express the state of equilibrium of mass and edge forces acting upon the half-strip.

For the symmetric problem, due to (1.4) equalities (5.11) and (5.13) become identities, so that (5.12)and (5.14) will serve as the conditions of damping.

For the skew-symmetric problem equalities (5.11) and (5.13) serve as conditions of damping, while (5.12) and (5.14) become identities.

6. We now turn to the question of procedure in the fulfillment of boundary conditions on the side surfaces of the plate in terms of arbitrary constants of the basic and the auxiliary iteration processes. For the sake of concreteness let us consider the following three variants of the boundary conditions for $\alpha = \alpha_0$ ($\xi = 0$)

$$\sigma_{\alpha\alpha} = 0, \qquad \sigma_{\alpha\beta} = 0, \qquad \sigma_{\alpha\gamma} = 0 \tag{6.1}$$

$$\sigma_{\alpha\alpha} = 0, \qquad \sigma_{\alpha\beta} = 0, \qquad W = 0 \tag{6.2}$$

$$u_\alpha = 0, \qquad u_\beta = 0, \qquad W = 0 \tag{6.3}$$

(they correspond to the free, hinge supported and rigidly built-in edges, respectively) and let us fulfill them, assuming that the total state of stress is composed of the basic state of stress determined by the basic iteration process, and of the boundary state of stress determined by the auxiliary iteration process. Let

$$\sigma_{\alpha\alpha} = h^{-q_1} \sum_{s=0}^{S} h^s \sigma_{\alpha\alpha}^{(s)} + h^{-q_2} \sum_{s=0}^{S} h^s\,(\sigma_{\alpha\alpha(1)}^{(s)} + \sigma_{\alpha\alpha(2)}^{(s)})$$
$$\sigma_{\alpha\beta} = h^{-q_1} \sum_{s=0}^{S} h^s \sigma_{\alpha\beta}^{(s)} + h^{-q_2} \sum_{s=0}^{S} h^s\,(\sigma_{\alpha\beta(1)}^{(s)} + \sigma_{\alpha\beta(2)}^{(s)}) \tag{6.4}$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$W = h^{-q_1}\sum_{s=0}^{S} h^s W^{(s)} + h^{-q_2}\sum_{s=0}^{S} h^s \left(W_{(1)}^{(s)} + W_{(2)}^{(s)}\right) \qquad (6.4)\ \text{cont.}$$

By q_1 we understand here the volumes of q determined by Formulas (2.3) and (2.4), which correspond to the basic iteration process, and by q_2 — the values of q determined by Formulas (5.2) which correspond to the auxiliary iteration process.

The number r remains undefined in Formulas (5.2). It should be so chosen that the procedure of imposing the boundary conditions would have recurrent character. For all three forms of boundary conditions considered, (6.1) through (6.3), this purpose is achieved if we set $r = 2$. Then, substituting (6.4) into (6.1) through (6.3) and setting the coefficients at all powers of h on the left-hand sides of resulting equalities equal to zero, we obtain the sequences of boundary conditions for the coefficients of expansions (6.4).

For $\alpha = \alpha_0$ ($\xi = 0$) they are written down as

$$\sigma_{\alpha\alpha}^{(s)} + \sigma_{\alpha\alpha(1)}^{(s)} + \sigma_{\alpha\alpha(2)}^{(s)} = 0, \qquad \sigma_{\alpha\beta}^{(s)} + \sigma_{\alpha\beta(1)}^{(s)} + \sigma_{\alpha\beta(2)}^{(s)} = 0$$
$$\sigma_{\alpha\gamma}^{(s-1)} + \sigma_{\alpha\gamma(1)}^{(s)} + \sigma_{\alpha\gamma(2)}^{(s)} = 0 \qquad (6.5)$$

$$\sigma_{\alpha\alpha}^{(s)} + \sigma_{\alpha\alpha(1)}^{(s)} + \sigma_{\alpha\alpha(2)}^{(s)} = 0, \qquad \sigma_{\alpha\beta}^{(s)} + \sigma_{\alpha\beta(1)}^{(s)} + \sigma_{\alpha\beta(2)}^{(s)} = 0$$
$$W^{(s)} + W_{(1)}^{(a)} + W_{(2)}^{(a)} = 0 \qquad (6.6)$$

$$u_\alpha^{(s)} + u_{\alpha(1)}^{(s-1)} + u_{\alpha(2)}^{(s-1)} = 0, \qquad u_\beta^{(s)} + u_{\beta(1)}^{(s-1)} + u_{\beta(2)}^{(s-1)} = 0$$
$$W^{(s)} + W_{(1)}^{(a)} + W_{(2)}^{(a)} = 0 \qquad (6.7)$$

where $a = s - 2$ for the skew-symmetric problem, and $a = s$ for the symmetric problem.

The boundary relations (6.5) through (6.7) contain quantities designated by additional subscripts (1) and (2). Their determination, as shown above, is reduced for each particular (s) to the integration of equations of torsion and plane strain problems. This implies that for each (s) the quantities contain sufficient number of arbitrary constants to fulfill all three boundary conditions formulated by equalities (6.5) through (6.7). However, only damped solutions are taken into consideration in the auxiliary iteration process, and hence the damping conditions (5.11) and (5.13) (for the skew-symmetric problem) or (5.12) and (5.14) (for the symmetric problem) must be fulfilled. These conditions are independent of the variable ζ, and therefore the arbitrary constants of the basic iteration process can be used in their fulfillment.

For each (s) in the symmetric or skew-symmetric case the construction of quantities $\sigma_{\alpha\alpha}^{(s)}, \sigma_{\alpha\beta}^{(s)}, \sigma_{\alpha\gamma}^{(s)}, u_\alpha^{(s)}, u_\beta^{(s)}, W^{(s)}$, contained in the conditions

(6.5) through (6.7) reduces to the integration of equations of the problems of plate bending (in the skew-symmetric case) or generalized plane stress (in the symmetric case). This furnishes a sufficient number of arbitrary constants for the fulfilment of all damping conditions.

Thus, the formal count indicates that the basic iteration process together with the auxiliary process give exactly the number of arbitrary constants required for the fulfilment of boundary conditions (6.5) through (6.7) and conditions of damping (5.11) through (5.14). A more detailed analysis of these conditions would take too much space. Such analysis was carried out in [1] for the problem of bending of a plate referred to a Cartesian coordinate system; it has shown that the procedure of fulfilment of boundary conditions has recurrent character.

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